Minimum Induced Drag of Ground Effect Wings

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Kida and Miyai's theory, which treats theoretically the problem of the minimum induced drag of nonplanar ground effect wings, is refined and extended. The gap clearance between the wing tip and the ground is assumed to be so small that the method of matched asymptotic expansions may be used. Except for this restriction, this method has a remarkable flexibility in wing front-view geometry. The theory is applied to some typical cases such as semielliptic, rectangular, and triangular ground effect wings over flat surface and a flat plate in a rectangular guide way. Special attention is concentrated on the two limiting cases with very high and low front views. Kida and Miyai's errors are corrected.

Nomenclature

$4a$	= span of vortex-trace in Ω plane
A_u, A_ℓ	= integral constants in outer solutions
B_u, B_ℓ	= [see Eqs. (5) and (6)]
$B(p,q)$	= beta function
C_0	= [see Eq. (12)]
E,K	= elliptic integrals of first and second kind
e	= span efficiency factor $= e_\epsilon + e_h$
e_ϵ, e_h	= span efficiency factor due to wing tip gap and midspan height
F	= complex velocity potential
$g_\alpha, G, G_\epsilon, G_\alpha$	= [see Eqs. (14)]
h	= ground height at midspan, normalized by semispan
$I_1, I_2, \text{---} I_7$	= integrals (see Appendix A)
J	= [see Eqs. (18)]
k	= modulus of Jacobi's elliptic function = $(1-k'^2)^{1/2}$
M	= [see Eq. (10)]
p,q	= real and imaginary parts of Ω
Q	= real part of $dZ/d\Omega$ on the vortex-trace
W_0	= downwash velocity at midspan in Trefftz plane
$Z=y+iz$	= independent variable in physical plane
$\pi\alpha$	= angle between wing tip and ground surface, measured in lower domain
β, β_T	= (see Figs. 1 and 2)
Γ	= spanwise bound vortex distribution
ϵ	= clearance between wing tips and ground surface
ζ_i	= (see Fig. 2)
$\Omega = p+iq$	= (see Fig. 1)
$(\ )^o, (\ )^{oi}$	= outer solution and its inner limit
$(\ )^i, (\ )^{io}$	= inner solution and its outer limit
$(\ )_u, (\ )_\ell$	= refer to upper and lower domains

I. Introduction

FOR several years, interest in the ground effect wing (GEW) for future high-speed overwater or railroad vehicles has increased. Ando,[1] and Mamada and Ando[2,3] presented theories on the minimum induced drag of semicircular and semielliptic front-view GEW's. Ashill[4] published an approximate theory for a rectangular GEW. Barrows[5] and Barrows and Widnall[6] treated a ram wing in a tube with different sections. Kida and Miyai[7] developed a theory, based on the matched asymptotic expansions, for GEW's having arbitrary front-view shapes with vanishing tip gaps.

In the present paper, the section shape of vortex sheet is assumed to be same as that of the wing front view, as in all previous papers. This may become true when both angle of attack and camber are reasonably small and in addition the spanwise lift distribution is optimum. Kida and Miyai's formulation of the general theory is substantially refined, and proved to be valid when the ground surface is curved. Applications are shown for some representative cases, such as semielliptic, rectangular, and triangular front-view GEW's over the flat ground, and a flat plate in a rectangular guide way. It is found that the span efficiency factor for each case consists of two additive major parts. The one is due to wing tip gap ϵ, and the other to height ratio h. When the wing tip is perpendicular to the ground, the former part always agrees with the span efficiency factor for the semicircular GEW, as pointed out for semielliptic GEW in Ref. 3. The result for the first example agrees with one obtained in Ref. 3. Kida and Miyai's expression of e_h for vanishing h is incorrect. The result for the second seems to agree with one given in Ref. 4; Kida and Miyai's e_h for vanishing h is again incorrect. The third result for vanishing h agrees with one obtained in Ref. 8 using the approximating *lateral nozzle model* technique, which may be valid for vanishing h. The fourth result agrees with one given in Ref. 5. It is interesting to note that, as stated by Barrows, e_h for the second and the fourth examples agree with the corresponding one for a flat wing over the flat ground.

Many useful numerical methods have been developed today for lifting surfaces. Nevertheless, analytical methods such as presented here have some significance. First, numerical methods of today are less suitable for drag estimation because of the significant leading edge force, which would require special considerations. Second, numerical methods might not illuminatingly show limiting properties for such cases as $\epsilon \to o$ and $h \to o$ or ∞, which are especially interesting from a theoretical viewpoint.

II. General Formulation

The general formulation used here resembles Kida and Miyai's approach, except for the following: a) The physical plane Z is mapped directly into the parameter plane Ω, not through the circle plane. This makes the theory substantially simpler and clearer. b) Treatment as a Riemann-Hilbert-Poincare problem by Kida and Miyai can be replaced by an elementary source-sink concept. c) The formulation is carried out for any curved ground surface explicitly, whereas Kida and Miyai did it merely for the flat ground surface.

A. Outer Solution

Upper and lower domains in the physical plane Z are separately mapped into the upper half domain of the parameter plane Ω with functions $Z_u(\Omega)$ and $Z_\ell(\Omega)$, respectively. (See Fig. 1.) Here, the upper domain is defined as an open region above the vortex trace, and the lower domain as a

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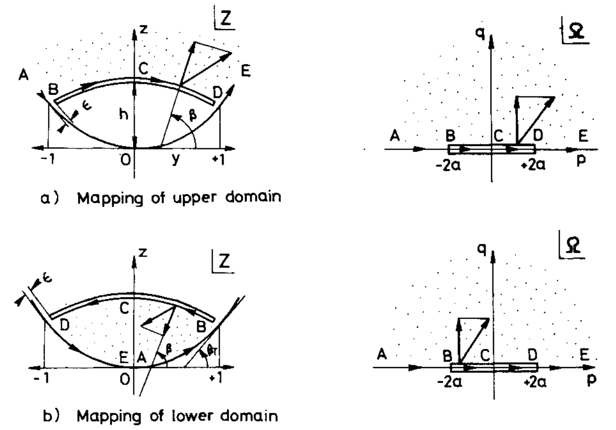

Fig. 1 Conformal mapping for outer solution.

closed region surrounded by the vortex trace and the ground surface. The lower half domain of the Ω plane is not necessary, but it is convenient to assume that the flow pattern in the Ω plane is symmetric about the real p-axis. For both mappings, the vortex trace corresponds to the same segment $-2a \leq p \leq +2a$ on the real axis of the Ω plane. Because of this symmetry about the real axis in the Ω plane, it is sufficient to place distributed sources or sinks on $-2a<p<2a$ and a pair of concentrated sources or sinks at $p=\pm 2a$. Then, on the remaining semi-infinite parts $|p|>2a$, corresponding to the ground surface, the boundary condition is automatically satisfied; namely the flow does not pass through these lines.

We treat only the case of optimum lift distribution throughout this paper, and thus the boundary condition on the vortex trace is specified by Munk's theorem. Writing it in the Ω plane leads to

$$\pm W_0 \, |dZ/d\Omega| \sin\beta = -\mathrm{Re}(idF/d\Omega) \qquad (1)$$

where $\pm$ refer to the upper and lower domains, respectively. Simple geometrical considerations allow $\sin\beta$ to be eliminated as follows

$$W_0 Q = -\mathrm{Re}(idF/d\Omega) \qquad (2a)$$

for

$$|p| \leq 2a \quad q=0 \qquad (2b)$$

where double signs disappear.

Our next step is to find the outer solutions of the complex velocity potential $F^o_{u,\ell}(\Omega)$, which satisfy the boundary condition [Eq. (2)]. For this purpose, source (sink) distributions in $-2a \leq p \leq 2a$ and a pair of concentrated sources (sinks) at $p=\pm 2a$ are assumed to be placed in the upper (lower) domain. Then, we have easily

$$\frac{dF^o}{d\Omega} = W_0 \left[-\frac{1}{\pi} \int_{-2a}^{2a} \frac{Q(p)}{\Omega - p} dp \pm \frac{2}{\pi} \frac{\Omega}{\Omega^2 - 4a^2} \right] \qquad (3)$$

Integrating Eq. (3) gives as the approximate outer solution of the complex velocity potential

$$F^o_u(\Omega) = \frac{W_0}{\pi} \left[\int_\Omega^{2a} d\Omega_1 \int_{-2a}^{2a} \frac{Q_u(p)}{\Omega_1 - p} dp + \ln(\Omega^2 - 4a^2) \right] + A_u \qquad (4a)$$

$$F^o_\ell(\Omega) = \frac{W_0}{\pi} \left[-\int_{-2a}^{\Omega} d\Omega_1 \int_{-2a}^{2a} \frac{Q_\ell(p)}{\Omega_1 - p} dp - \ln(\Omega^2 - 4a^2) \right] + A_\ell \qquad (4b)$$

The constants of integration, A_u, A_ℓ are determined by matching with the inner solutions.

Now we derive expressions for the inner limit of the outer solutions, for later use. Considering the angular relations between the $Z-$ and $\Omega-$ planes at the wing tip $Z=Z_T$ gives

$$(Z_u - Z_T)e^{-i\beta_T} = B_u(\Omega - 2a)^{1-\alpha} \qquad (5)$$

$$(Z_\ell - z_T)e^{-i\beta_T} = B_\ell(\Omega + 2a)^{\alpha} \qquad (6)$$

from which

$$B_u e^{i\beta_T} = \lim_{\Omega \to 2a} \left[\frac{(\Omega - 2a)^\alpha}{1-\alpha} Z'_u(\Omega) \right] \qquad (7a)$$

$$B_\ell e^{i\beta_T} = \lim_{\Omega \to -2a} \left[\frac{(\Omega + 2a)^{1-\alpha}}{\alpha} Z'_\ell(\Omega) \right] \qquad (7b)$$

Letting $Z \to Z_T$, $(\Omega \to \pm 2a)$ in Eq. (4) makes the double integral terms vanish. In the remaining terms, $(\Omega^2 - 4a^2)$ is eliminated by using Eqs. (5) and (6) to give

$$F^{oi}_u = \frac{W_0}{\pi} \left[\ln 4a + \frac{1}{1-\alpha} \ln \frac{Z_u - Z_T}{B_u} - \frac{i\beta_T}{1-\alpha} \right] + A_u \qquad (8a)$$

$$F^{oi}_\ell = \frac{W_0}{\pi} \left[-\ln 4a - \frac{1}{\alpha} \ln \frac{Z_\ell - Z_T}{B_\ell} + \frac{i\beta_T}{\alpha} \right] + A_\ell \qquad (8b)$$

B. Inner Solution

Now the following may be introduced as the inner variable:

$$Z_i \equiv [(Z - Z_T)/\epsilon] e^{-i\beta_T} \qquad (9)$$

Then, it follows that the inner solution of the velocity potential (see Fig. 2) may be given by

$$F^i(Z_i) = (M/\pi) \ln \zeta_i \qquad (10)$$

$$Z_i = C_0 [\zeta_i^{1-\alpha}/(1-\alpha) - \zeta_i^{-\alpha}/\alpha] \qquad (11)$$

$$C_0 = \alpha(1-\alpha)/\sin \pi\alpha \qquad (12)$$

The outer limit of the inner solution is obtained as follows:

$$F^{io}_u = (M/\pi) \lim_{\xi \to \infty} \ln \xi_i$$
$$= \frac{M}{\pi} \frac{1}{1-\alpha} \left[\ln \frac{Z_u - Z_T}{\epsilon} - \ln \frac{C_0}{1-\alpha} - i\beta_T \right] \qquad (13a)$$

$$F^{oi}_\ell = \frac{M}{\pi} \lim_{\xi_i \to o} \ln \xi_i$$
$$= \frac{M}{\pi} \frac{-1}{\alpha} \left[\ln \frac{Z_\ell - Z_T}{\epsilon} - \ln \frac{C_0}{\alpha} - i\beta_T \right] \qquad (13b)$$

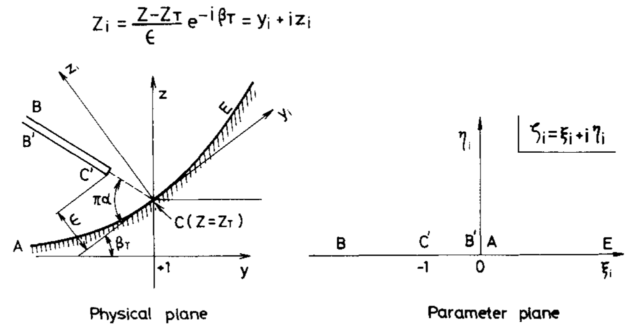

Fig. 2 Conformal mapping for inner solution.

C. Matching Procedure

According to the principle of matched asymptotic expansions, we put $F^{oi}_{u,\ell}=F^{io}_{u,\ell}$ using Eqs. (8) and (13). Then we have $W_0=M$ and

$$\mathrm{Re}(A_u-A_\ell)=(W_0/\pi)[-2\ln 4a+G] \qquad (14a)$$

$$G\equiv G_\alpha+G_\epsilon \quad G_\alpha\equiv \ln g_\alpha \quad G_\epsilon\equiv[1/\alpha(1-\alpha)]\ln 1/\epsilon \qquad (14b)$$

$$g_\alpha\equiv[((1-\alpha)/C_0)\,|B_u|]^{1/(1/\alpha)}[(\alpha/C_0)\,|B_\ell|]^{1/\alpha} \qquad (14c)$$

It is noteworthy that β_T may be eliminated throughout in Eqs. (14) because only the absolute values of B_u and B_ℓ are required.

D. Expression of Span Efficiency Factor

Span efficiency factor e may be calculated for cases of the optimum lift distributions by

$$e=\frac{1}{\pi W_0}\int_{-1}^{1}\Gamma(y)dy \qquad (15)$$

Here $\Gamma(y)$, the spanwise bound vortex distribution, must be the composite solution Γ^c

$$\Gamma=\mathrm{Re}(F^c_u-F^c_\ell)=\Gamma^c=\Gamma^o+\Gamma^i-\Gamma^{oi} \qquad (16)$$

Careful examination shows that, although $(\Gamma^i-\Gamma^{oi})$ becomes infinite logarithmically at the wing tips, the integral over the span is negligible (Kida and Miyai made a trivial error in this regard): Thus, Γ in Eq. (15) may be replaced by Γ^o only. It is instructive to point out that the inner solution does not appear explicitly but provides its influence implicitly through Re (A_u-A_ℓ), the integration constants of the outer solutions, which becomes quite important for vanishing tip gaps.

Some appropriate substitutions in e and later manipulations give the following result:

$$e=(1/\pi^2)[J+2G(\epsilon,\alpha)-4\ln 4a] \qquad (17)$$

$$J=J_1+J_2 \quad J_i=J_{iu}+J_{i\ell} \quad i=1,2 \qquad (18a)$$

$$J_{1u}/4=\int_0^{2a}Q_u(p)\,\ln|p^2-4a^2|dp \qquad (18b)$$

$$J_{1\ell}/4=-\int_0^{2a}Q_\ell(p)\,\ln|\,p^2-4a^2|dp \qquad (18c)$$

$$J_{2u}/(-2)=\int\int_0^{2a}Q_u(p_1)Q_u(p_2)\,\ln|p_1^2-p_2^2|dp_1dp_2 \qquad (18d)$$

$$J_{2\ell}/(-2)=\int\int_0^{2a}Q_\ell(p_1)Q_\ell(p_2)\,\ln|p_1^2-p_2^2|dp_1dp_2 \qquad (18e)$$

It may again be noted that β_T is not shown explicitly in the above equations because G contains B_u and B_ℓ as their absolute values only.

III. Numerical Examples

A. Semielliptic Ground Effect Wing $(0<h<1)$

We use the following mappings (Fig. 3) $Z_u\rightarrow W\rightarrow\Omega$, and $Z_\ell\rightarrow W\rightarrow\Omega=\mathrm{sn}t$:

$$Z_{u,\ell}=C\sin w\begin{cases}C\sin(\sin^{-1}\Omega+i\eta_0)\\ C\sin(-C'_\ell k_\ell t+C''_\ell)\end{cases} \qquad (19)$$

$$C=1/\cosh\eta_0 \quad C'_\ell=\pi/(2K_\ell K_\ell) \quad C''_\ell=i\eta_0 \qquad (20)$$

$$h=\tanh\eta_0=\tanh(\pi K'_\ell/2K_\ell) \qquad (21)$$

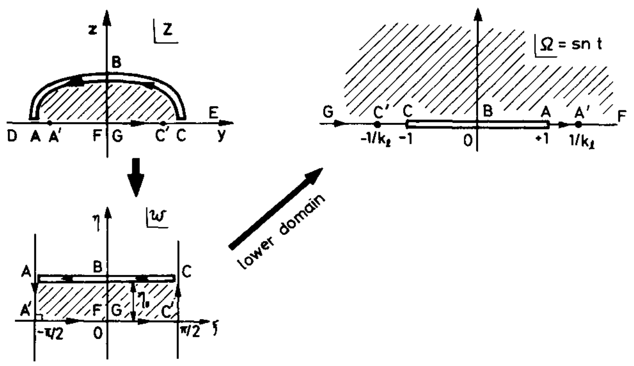

Fig. 3 Conformal mapping for semielliptic ground effect wing.

Then we have

$$Q_u(p)=\mathrm{Re}(dZ_u/d\Omega)=1 \qquad (22a)$$

$$Q_\ell(p)=-(\pi/2K_\ell)\cos(\pi t/2K_\ell)/\mathrm{cn}t\ \mathrm{dn}t \qquad (22b)$$

$$B_u=\sqrt{2}\tanh(\pi K'_\ell/2K_\ell) \qquad (23a)$$

$$B_\ell=i\pi\tanh\eta_0/\sqrt{2}k'_\ell K_\ell \qquad (23b)$$

$$J_{1u}/4=\int_0^1\ln|p^2-1|dp=2\ln 2-2 \qquad (24)$$

$$J_{2u}/(-2)=\int\int_0^1\ln|p_1^2-p_2^2|dp_1dp_2=2\ln 2-3 \qquad (25)$$

$$J_{1\ell}/(-4)=-\int_0^{\pi/2}\cos\rho\,\ln\left|\mathrm{sn}^2\left[\frac{2K_\ell}{\pi}\rho\right]-1\right|d\rho \qquad (26)$$

$$J_{2\ell}/(-2)=\int\int_0^{\pi/2}\cos\rho_1\cos\rho_2\,\ln\left|\mathrm{sn}^2\left[\frac{2K_\ell}{\pi}\rho_1\right]-\mathrm{sn}^2\left[\frac{2K_\ell}{\pi}\rho_2\right]\right|d\rho_1d\rho_2 \qquad (27)$$

The expression e for the general values of $h<1$ may be obtained as in Ref. 7 (see Appendix B). For the limiting case of $h\ll 1$, we have

$$k_\ell\rightarrow 1 \quad K_\ell\simeq\pi^2/4h\rightarrow\infty \qquad (28a)$$

$$J_{1\ell}\simeq(16/\pi-8)K_\ell \quad J_{2\ell}\simeq(6-16/\pi)K_\ell \qquad (28b)$$

$$J\simeq-2K_\ell \quad g_\alpha\simeq 2^4\pi^2h^4/(k'_\ell K_\ell)^2 \qquad (28c)$$

Here $J_{1\ell}$ and $J_{2\ell}$ are obtained by using sn $u\sim\tanh u$ and the series expressions

$$\tanh x=1+2\sum_{n=1}^{\infty}(-1)^n e^{-2nx} \quad \coth x=1+2\sum_{n=1}^{\infty}e^{-2nx} \qquad (29)$$

for $x>0$

Thus, e is obtained as

$$e-(8/\pi^2)\ln(1/\epsilon)\simeq \tfrac{1}{2}h\epsilon \ll h\ll 1 \qquad (30)$$

which agrees with the result in Ref. 3. On the other hand, Kida and Miyai[7] gave $1/h$ instead of $\frac{1}{2}h$. They estimated the value of J as a power series of $q=(1-h)/(1+h)$, which diverges as q tends to unity $(k\rightarrow 1)$; thus, they did not use the correct value of J. Moreover, it must be noted that, when the mapping [Eq. (19)] is used the values of $k_\ell=0\sim 1$ correspond to $h=1\sim 0$, and thus the $h>1$ case cannot be treated. Therefore, the interesting feature,[3] e is symmetric in exchanging $h\leftrightarrow 1/h$, may not be proved from the present mappings.

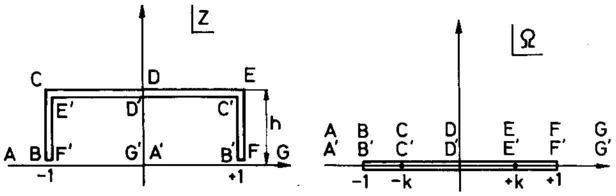

Fig. 4 Mapping for rectangular ground effect wing.

B. Rectangular Ground Effect Wing

Mappings of $Z_{u,\ell} \to \Omega \to t$ (Fig. 4) give

$$Z_u = C_u[E(t) - k_u'^2 t] + ih \quad \Omega \equiv k_u \mathrm{sn}t \tag{31a}$$

$$Z_\ell = -C_\ell t + ih \quad \Omega \equiv k_\ell \mathrm{sn}t \tag{31b}$$

$$C_u = 1/[E_u - k_u'^2 K_u] \quad C_\ell = 1/K_\ell \tag{32a}$$

$$h = \frac{k_u'^2 K_u' + (E_u' - K_u')}{E_u - k_u'^2 K_u} = K_\ell'/K_\ell \tag{32b}$$

$$E_u' \equiv E(K_u') \quad K_\ell' \equiv K(k_\ell'), \text{ etc} \tag{32c}$$

Then we have

$$B_u = \sqrt{2} k_u'/(E_u - k_u'^2 K_u) \quad B_\ell = \sqrt{2} i/k_\ell' K_\ell \tag{33a}$$

$$Q_u = C_u k_u \, \mathrm{cn}t/\mathrm{dn}t \quad (\partial p| \le k_u) \tag{33b}$$

$$Q_\ell = -C_\ell/(k_\ell \, \mathrm{cn}t \, \mathrm{dn}t) \quad (|p| \le k_\ell) \tag{33c}$$

$$J_{1u}/4 = C_u k_u^2 \int_0^{K_u} \mathrm{cn}^2 t \ln \mathrm{dn}^2 t dt \tag{34a}$$

$$J_{1\ell}/(-4) = -C_\ell \int_0^{K_\ell} \ln \mathrm{dn}^2 t dt \tag{34b}$$

$$J_{2\ell}/(-2) = C_\ell^2 [K_\ell^2 \ln k_\ell^2 + I_2(K_\ell)] \tag{34c}$$

$$J_{2u}/(-2) = C_u^2 k_u^4 \left\{ \ln k_u^2 \left[\int_0^{K_u} \mathrm{cn}^2 t \, dt \right]^2 + I_1(K_u) \right\} \tag{34d}$$

J_{1u} and $J_{1\ell}$ are evaluated by using formulas in Ref. 10. The integrals I_1 and I_2 are defined in Appendix A. For $h \ll 1$ we have

$$k_u, k_\ell \to 1 \tag{35a}$$

$$h = (\pi/4) k_u'^2 = \pi/2K_\ell \tag{35b}$$

$$J_{1u}/4 \simeq 2 \ln 2 - 2 \quad J_{1\ell}/4 \simeq -\pi/2h \tag{35c}$$

$$J_{2u} \simeq 2(3 - 2\ln 2) \quad J_{2\ell} \simeq (4/3) K_\ell \quad J \simeq -4\pi/3h \tag{35d}$$

$$g_\alpha \simeq (64/\pi^3) h^3 \exp(\pi/h) \tag{35e}$$

J_{2u} and $J_{2\ell}$ are estimated as for the semielliptic case. Thus, we have

$$e - (8/\pi^2) \ln(1/\epsilon) \simeq \tfrac{2}{3}\pi h \epsilon \ll h \ll 1 \tag{36}$$

It is interesting to note that the right-hand side agrees approximately with one for a straight wing over the flat ground surface, as obtained by Barrows.[6] Kida and Miyai[7] gave $2/\pi h$ instead of $\frac{2}{3}\pi h$, being derived by an incorrect estimation of J.

For another limiting case $h \gg 1$, we have

$$k_u, k_\ell \to 0 \quad h \simeq 4/\pi k_u^2 \tag{37a}$$

$$J_{1u} \simeq 0(1) \quad J_{1\ell} \simeq -0 \quad J_{2u} \simeq 2 \ln h \quad J_{2\ell} \simeq 2\pi h \tag{37b}$$

and

$$e - (8/\pi^2) \ln(1/\epsilon) \simeq 2h/\pi \quad h \gg 1 \tag{38}$$

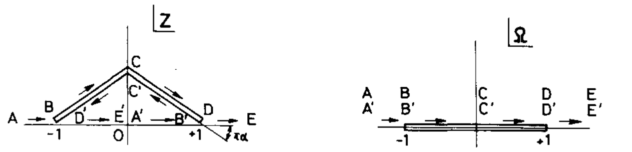

Fig. 5 Mapping for triangular ground effect wing.

Here, J_{2u} and $J_{2\ell}$ are estimated by using $\mathrm{sn}u \sim \sin u$, $\mathrm{cn}u \sim \cos u$; and series expansion

$$\ln|\cos\zeta_1 - \cos\zeta_2| = -\ln 2 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \cos n\zeta \cos n\zeta_2 \tag{39}$$

Kida and Miyai[7] have not treated the last case.

C. Triangular Ground Effect Wing

The Schwarz-Christoffel transform (Fig. 5) gives

$$Z_u = C_u \int_0^\Omega \Omega^{2\alpha} (\Omega^2 - 1)^{-\alpha} d\Omega + ih \tag{40a}$$

$$Z_\ell = C_\ell \int_0^\Omega \Omega^{-2\alpha} (\Omega^2 - 1)^{-1+\alpha} d\Omega + ih \tag{40b}$$

$$C_u = (1 - ih) \exp(i\pi\alpha)/I_u(\alpha) \tag{40c}$$

$$C_\ell = -(1 + ih) \exp[i\pi(1-\alpha)]/I_\ell(\alpha) \tag{40d}$$

$$I_u(\alpha) = \tfrac{1}{2} \cdot B(\alpha + \tfrac{1}{2}, 1 - \alpha) \tag{40e}$$

$$I_\ell(\alpha) = \tfrac{1}{2} \cdot B(\tfrac{1}{2} - \alpha, \alpha) \tag{40f}$$

where B is the beta function. Then we have

$$|B_u| = (1+h^2)^{1/2}/[2^\alpha (1-\alpha) I_u(\alpha)] \tag{41a}$$

$$|B_\ell| = (1+h^2)^{1/2}/[2^{1-\alpha} \cdot \alpha \cdot I_\ell(\alpha)] \tag{41b}$$

$$Q_u(p) = p^{2\alpha} (1-p^2)^{-\alpha}/I_u(\alpha) \tag{42a}$$

$$Q_\ell(p) = -p^{-2\alpha} (1-p^2)^{\alpha-1}/I_\ell(\alpha) \tag{42b}$$

$$J_{1u} = [4/I_u(\alpha)] I_3 \tag{43a}$$

$$J_{1\ell} = [4/I_\ell(\alpha)] I_4 \tag{43b}$$

$$J_{2u} = [-2/I_u^2(\alpha)] I_5 \tag{43c}$$

$$J_{2\ell} = [-2/I_\ell^2(\alpha)] I_6 \tag{43d}$$

Estimations of the integrals I_3,----I_6 are described in the Appendix A. When $h \ll 1$ but $\epsilon \ll \alpha$, we have

$$h \doteqdot \pi\alpha \quad I_u(\alpha) \simeq 1 \quad I_\ell(\alpha) \simeq 1/2\alpha \tag{44a}$$

$$|B_\ell| \simeq 2^\alpha \quad G \simeq (1/\alpha) \ln(\pi\alpha/\epsilon) \tag{44b}$$

$$J_{1u} \simeq 8(\ln 2 - 1) \quad J_{1\ell} \simeq 4(2 \ln 2 - 1/\alpha) \tag{44c}$$

$$J_{2u} \simeq -2(2 \ln 2 - 3) \quad J_{2\ell} \simeq 1/\alpha \tag{44d}$$

and

$$e = (2/\pi^2 \alpha)[\ln(\pi\alpha/\epsilon) - 3/2] \quad \epsilon \ll \alpha \ll 1 \tag{45}$$

On the other hand, when $h \gg 1$, we have

$$\alpha' \equiv \tfrac{1}{2} - \alpha \to 0 \quad h \simeq 1/\pi\alpha' \tag{46a}$$

$$I_u(\alpha) \simeq 1 \quad I_\ell(\alpha) \simeq 1/2\alpha' \tag{46b}$$

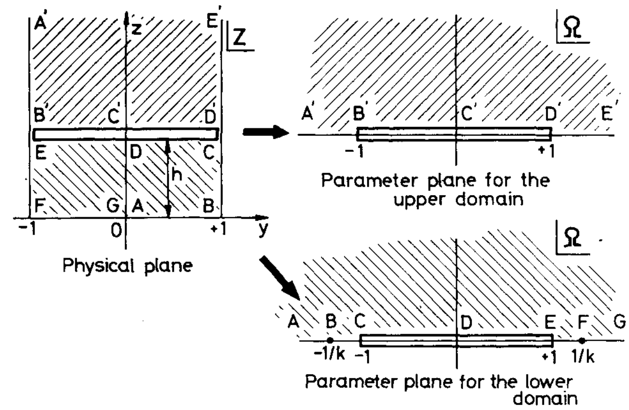

Fig. 6 Mapping for flat plate in rectangular guide way.

$$|B_u| \simeq \sqrt{2}h \quad |B_\ell| \simeq \sqrt{8}/\pi \tag{46c}$$

$$G \simeq 4\ \ln(1/\epsilon) + 2\ \ln(1/\alpha') \tag{46d}$$

$$J_{1u} \simeq -16 \quad J_{1\ell} \simeq -2\pi^2\alpha' \sim 0 \tag{46e}$$

$$J_{2u} \simeq 6(1-\ln 2) \quad J_{2\ell} \simeq 1/\alpha' \tag{46f}$$

Then we obtain

$$e - (8/\pi^2)\ln(1/\epsilon) \simeq (h/\pi) \quad h \gg 1 \tag{47}$$

Equation (45) agrees with that of Ref. 8, which was obtained by the present author et al. by Barrows and Widnall's technique.[6]

D. Flat Plate in Rectangular Guide Way

We use following conformal mappings (Fig. 6):

$$Z_u = C_u \sin^{-1}\Omega + ih \tag{48a}$$

$$Z_\ell = -kC_\ell t + ih \quad \Omega \equiv \mathrm{sn}t \tag{48b}$$

$$C_u = 2/\pi \quad C_\ell = 1/kK \tag{48c}$$

Then we have

$$h = K'/K \tag{49a}$$

$$Q_u = (2/\pi)/(1-p^2)^{1/2} \quad Q_\ell = -1/(K\ \mathrm{cn}t\ \mathrm{dn}t) \tag{49b}$$

$$|B_u| = 4/\sqrt{2}\pi \quad |B_\ell| = \sqrt{2}/k'K \tag{49c}$$

$$g_\alpha = (16/\pi k'K)^2 \quad G_\epsilon = 4\ \ln(1/\epsilon) \tag{49d}$$

$$J_{1u}/4 = -\ln 2 \tag{50a}$$

$$J_{1\ell}/(-4) = (\pi K'/2K) - 2\ \ln(k'/k) \tag{50b}$$

$$J_{2u}/(-2) = -2\ \ln 2 \tag{50c}$$

$$J_{2\ell}/(-2) = (1/K^2)\cdot I_2(K) \tag{50d}$$

The $J_{2\ell}$ integral appears in Sec. III B. For $h \ll 1$, we have

$$k \to 1 \quad h \simeq \pi/2K \quad C_\ell \simeq 1/K \tag{51a}$$

$$J_{1\ell} \simeq -4K \quad J_{2\ell} \simeq 4K/3 \tag{51b}$$

$$e - (8/\pi^2)\ \ln(1/\epsilon) \simeq \tfrac{2}{3}\pi h \quad \epsilon \ll h \ll 1 \tag{52}$$

The Equation (52) agrees with one given in Ref. 5. For $h \gg 1$, we have

$$k \to 0 \quad g_\alpha \simeq (32/\pi^2)^2 = \text{finite} \tag{53a}$$

$$J \sim \text{finite} \tag{53b}$$

and

$$e \simeq (8/\pi^2)\ \ln(1/\epsilon) \quad h \gg 1 \tag{54}$$

Namely, the result [Eq. (54)] refers to either straight wings arranged side by side infinitely in an unbounded flowfield, or a straight wing placed between and in close proximity to two vertical infinite walls. Here h does not contribute to e.

IV. Conclusions

General formulations are substantially simplified, revised, and extended, in comparison with Ref. 7. The numerical examples show the span efficiency factor e to be written in an additive form $e = e_\epsilon + e_h$, where e_ϵ and e_h result from the effects due to tip gaps e and height ratio h, respectively. Their limiting cases $h \ll 1$ and $h \gg 1$ agree with ones obtained by different previous methods, when available. Errors in Ref. 7 are corrected by proper estimation of certain double integrals. It is interesting to note that three cases, rectangular and flat wings on the flat ground and a flat wing in a rectangular channel have the same e_h for vanishing h.

Appendix A Estimation of Some Double Integrals

Here a few double integrals will be evaluated as typical examples.

$$1)\ I_1 \equiv \int\int_0^K \mathrm{cn}^2 t_1\ \mathrm{cn}^2 t_2 \ln|\mathrm{sn}^2 t_1 - \mathrm{sn}^2 t_2|\, dt_1 dt_2 \tag{A1}$$

Putting $\mathrm{sn}t_1 = \sin\varphi_1$, etc. and using Eq. (39) give

$$I_1 = -2\ \ln 2\left[\int_0^{\pi/2} \frac{\cos^2\varphi d\varphi}{(1-k^2\sin^2\varphi)^{1/2}} \equiv I_{11}\right]^2$$

$$-2\sum_{n=1}^{\infty}\frac{1}{n}\left[\int_0^{\pi/2}\frac{\cos^2\varphi\ \cos 2n\varphi d\varphi}{(1-k^2\sin^2\varphi)^{1/2}} \equiv I_{12}\right]^2 \tag{A2}$$

For $k \to 0$

$$I_{11} \simeq \pi/4 \quad I_{12} \simeq \tfrac{1}{2}\int_0^{\pi/2} \cos 2\varphi\ \cos 2n\varphi = 0 \tag{A3}$$

Hence

$$I_1 \simeq -(\pi^2/8)\ \ln 2 \tag{A4}$$

For $k \to 1$

$$I_{11} \simeq 1 \quad I_{12} \simeq (-1)^{n-1}/(4n^2-1) \tag{A5}$$

Hence

$$I_1 \simeq 2\ \ln 2 - 3 \tag{A6}$$

$$2)\ I_2 \equiv \int\int_0^K \ln|\mathrm{sn}^2 t_1 - \mathrm{sn}^2 t_2|\, dt_1 dt_2 \tag{A7}$$

For this integral, the above technique is also applicable for $k \simeq 0$ and gives

$$I_2 \simeq -(\pi^2/2)\ln 2 \tag{A8}$$

For another limit $k \simeq 1$, however, the same technique is not applicable. Hence, the method described in the text is used.

$$3)\ I_3 \equiv \int_0^1 p^{2\alpha}(1-p^2)^{-\alpha}\ln|1-p^2|\, dp \tag{A9}$$

$$I_4 \equiv \int_0^1 p^{-2\alpha}(1-p^2)^{\alpha-1}\ln|1-p^2|dp \qquad \text{(A10)}$$

These integrals are evaluated by putting $p^2=1-x$ and using the formula[9]

$$\int_0^1 x^{\mu-1}(1-x^r)^{\nu-1}\ln x dx$$

$$=(1/r^2)B(\mu/r,\nu)[\psi(\mu/r)-\psi(\nu+\mu/r)] \qquad \text{(A11)}$$

where B and ψ represent beta and psi functions.

4) $$I_5 \equiv \int\int_0^1 (p_1p_2)^{2\alpha}][(1-p_1^2)(1-p_2^2)]^{-\alpha}$$

$$\times \ln|p_1^2-p_2^2|dp_1dp_2 \qquad \text{(A12)}$$

$$I_6 \equiv \int\int_0^1 (p_1p_2)^{-2\alpha}[(1-p_1^2)(1-p_2^2)]^{\alpha-1}$$

$$\times \ln|p_1^2-p_2^2|dp_1dp_2 \qquad \text{(A13)}$$

These integrals are generalized to

$$I_7(p,q) \equiv \int\int_0^1 (x\cdot y)^{p-1}[(1-x)(1-y)]^{q-1}$$

$$\times \ln|x-y|dxdy \quad 0<p,q<1 \qquad \text{(A14)}$$

Putting $x \equiv \cos^2\theta$, etc. and using Eq. (39) lead to

$$I_7(p,q) = -2[\ln 2+K(p,q)]B^2(p,q) \qquad \text{(A15)}$$

where

$$K(p,q) \equiv \sum_{n=1}^{\infty} \frac{1}{n}\left[\sum_{r=0}^{n}(-1)^r \binom{2n}{2} \times \frac{\prod_{s=0}^{n-r-1}(p+s)\prod_{s=0}^{r-1}(q+s)}{\prod_{s=0}^{n-1}(p+q+s)}\right]^2 \qquad \text{(A16)}$$

Numerical calculations with a high-speed digital computer show that taking $n=10$ converges K satisfactorily if both p and q are larger than, say, 0.1.

Fortunately I_5 for $\alpha \to 0$ may be evaluated straightforward, i.e.

$$\lim_{\alpha\to 0} I_5 = \int\int_0^1 \ln|p_1^2-p_2^2|dp_1dp_2 = 2\ln 2-3 \qquad \text{(A17)}$$

I_6, however, can not be evaluated straightforwardly. Use of Eq. (A15) is also not successful, because in this case $p \to ½$ and $q \to 0$ as $\alpha \to 0$ and thus Eq. (A16) converges unpractically slowly.

Thus, another method should be developed to evaluate $I_6(\alpha)$ for vanishing α.

$$\lim_{\alpha\to 0} I_6(\alpha) = \hat{I}_6(\alpha)$$

$$= \int\int_0^1 \frac{\ln|p_1^2-p_2^2|dp_1dp_2}{(p_1p_2)^{1-2\alpha}[(1-p_1^2)(1-p_2^2)]^{½}} \qquad \text{(A18)}$$

The integral domain is divided into four parts, as shown in Fig. 7, where we assume

$$0<\alpha \ll \delta_1 < \delta \ll 1 \qquad \text{(A19)}$$

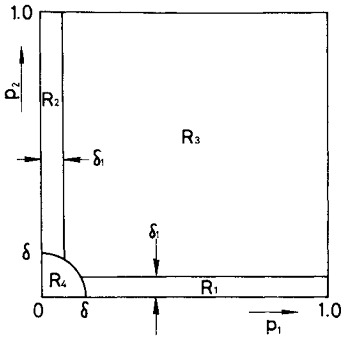

Fig. 7 Division of integral domain for $I_6(\alpha)$.

Then, it is seen that there are four contributions:

from $\quad R_1, R_2 = 0[(1/\alpha\delta_1)\ln\delta_1]$ (A20a)

from $\quad R_3 = 0(1/\delta_1^2)$ (A20b)

The remaining contribution from R_4 may be evaluated by putting

$$p_1 = \gamma\cos\theta \quad p_2 = \gamma\sin\theta \qquad \text{(A21a)}$$

$$0 \le \gamma \le \delta \ll 1 \quad 0 \le \theta \le \pi/2 \qquad \text{(A21b)}$$

and using Eq. (A11) with $r=2$ and $x=\cos\theta$, the contribution from

$$R_4 = -1/8\alpha^3 \qquad \text{(A22)}$$

Thus we have

$$I_6 = \hat{I}_6 \simeq -1/8\alpha^3 \text{ for } \alpha \to 0 \qquad \text{(A23)}$$

Appendix B e_h for a Semielliptic Ground Effect Wing

Mamada and Ando[3] gave e_h for a semielliptic GEW with vanishing tip gap

$$e_h = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\frac{16n^3}{(4n^2-1)^2} \qquad q \equiv \frac{1-h}{1+h} \qquad \text{(B1)}$$

Kida and Miyai[7] obtained

$$e_h = \frac{8}{\pi^2}\left[\frac{16}{9}q^2 + \left(\frac{16}{9}-\frac{\pi^2}{16}\right)q^4 + \left(\frac{277}{90}-\frac{\pi}{20}\right)q^6 + \left(\frac{65}{36}-\frac{\pi^2}{16}\right)q^8 + O(q^{10})\right] \qquad \text{(B2)}$$

We examined Kida and Miyai's procedure carefully and found Eq. (B2) to be imperfect. Instead, we obtained the following expression:

$$e_h = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{4n}} \times \left[\frac{16}{9} - \sum_{m=1}^{\infty} q^{2nm}\frac{32(16m^6-40m^4+29m^2-9)}{(4m^2-9)^2(4m^2-1)^2}\right] \qquad \text{(B3)}$$

Numerical comparison shows that Eqs. (B1) and (B3) are essentially identical for a wide range of h, but (B2) underestimates e_h as h decreases. For example, Eq. (B1), (B2), and (B3) give $e_h=0.639$, 0.561, and 0.640 respectively, for $h=0.3$.

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